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Conductivity of a Relativistic Plasma

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Abstract

The collision operator for a relativistic plasma is reformulated in terms of an expansion in spherical harmonics. In this formulation the collision operator is expressed in terms of five scalar potentials which are given by one-dimensional integrals over the distribution function. This formulation is used to calculate the electrical conductivity of a uniform electron-ion plasma with infinitely massive ions.

I. Introduction

Landau[1] first obtained an accurate kinetic equation for a nonrelativistic plasma. The Landau collision operator was generalized to a relativistic plasma by Beliaev and Budker.[2] The collision operator in both cases involves integrals of the distribution function of the background species. This makes the operators difficult to evaluate, numerically or analytically.

In the nonrelativistic case this difficulty was removed by Rosenbluth, MacDonald, and Judd,[3] and by Trubnikov.[4] They recast the Landau operator into differential form by writing it in terms of derivatives of two scalar potentials. The potentials in turn satisfy a pair of elliptic partial differential equations. With the aid of this formulation, the numerical evaluation of the collision operator may be accomplished straightforwardly by solving the potential equations. An analytical solution of these equations in terms of spherical harmonics was given by Rosenbluth *et al*.[3]

Recently,[5] we formulated the relativistic collision operator of Beliaev and Budker[2] in terms of five scalar potentials, which again obey elliptic partial differential equations. In the present paper we extend this formulation by solving the potential equations in terms of an expansion in spherical harmonics. As an application, we calculate the electrical conductivity of a relativistic plasma with massive ions.

In Sec. II we review the differential formulation of the collision operator. The spherical harmonic expansion is developed in Sec. III. It is shown in Sec. IV how the nonrelativistic results are recovered in the limit $c \to \infty$. We evaluate the zeroth-order and first-order spherical harmonic components of the collision term with respect to a Maxwellian background in Secs. V and VI. The calculation of the conductivity is given in Sec. VII.

II. Potentials

We begin by summarizing the differential formulation of the collision operator. This repeats our earlier exposition[5]; however, we now introduce a more compact notation and also write the results in such a way that the nonrelativistic limit is more easily obtained.

The collision term for a plasma of species s colliding off species s' may be written in the Fokker-Planck form as[1,2]:

$$C^{s/s'}(f_s, f_{s'}) = \frac{\partial}{\partial \mathbf{u}} \cdot \left(\mathsf{D}^{s/s'} \cdot \frac{\partial f_s}{\partial \mathbf{u}} - \mathbf{F}^{s/s'} f_s \right), \tag{1}$$

where the diffusion and friction coefficients are given by

$$\mathsf{D}^{s/s'}(\mathbf{u}) = \frac{\Gamma^{s/s'}}{2n_{s'}} \int \mathsf{U}(\mathbf{u},\mathbf{u}') f_{s'}(\mathbf{u}')\, d^3\mathbf{u}', \tag{2a}$$

$$\mathbf{F}^{s/s'}(\mathbf{u}) = -\frac{\Gamma^{s/s'}}{2n_{s'}} \frac{m_s}{m_{s'}} \int \left(\frac{\partial}{\partial \mathbf{u}'} \cdot \mathsf{U}(\mathbf{u},\mathbf{u}')\right) f_{s'}(\mathbf{u}')\, d^3\mathbf{u}', \tag{2b}$$

$$\Gamma^{s/s'} = \frac{n_{s'} q_s^2 q_{s'}^2 \log \Lambda^{s/s'}}{4\pi\epsilon_0^2 m_s^2}.$$

Here, $\mathbf{u}$ is the momentum per unit rest mass, and, in the relativistic case, the kernel U is given by[2]

$$\mathsf{U}(\mathbf{u},\mathbf{u}') = \frac{r^2}{\gamma\gamma' w^3}\Big(w^2 \mathsf{I} - \mathbf{u}\mathbf{u} - \mathbf{u}'\mathbf{u}' + r(\mathbf{u}\mathbf{u}' + \mathbf{u}'\mathbf{u})\Big), \tag{3}$$

in which $\gamma = \sqrt{1 + \mathbf{u}^2/c^2}$, $\gamma' = \sqrt{1 + \mathbf{u}'^2/c^2}$, and

$$r = \gamma\gamma' - \mathbf{u} \cdot \mathbf{u}'/c^2, \tag{4a}$$

$$w = c\sqrt{r^2 - 1}. \tag{4b}$$

The quantity r is the relativistic correction factor corresponding to the relative velocity of the two interacting particles. The relative speed of the interacting particles is given by w/r. In the nonrelativistic limit, $r \to 1$ and $w \to |\mathbf{u} - \mathbf{u}'|$.

In the previous paper[5] we expressed the Eqs. (2) in differential form, making use of the potentials

$$\Psi_{s'[1]0}(\mathbf{u}) = -\frac{1}{4\pi} \int w^{-1} f_{s'}(\mathbf{u}') \frac{d^3\mathbf{u}'}{\gamma'}, \tag{5a}$$

$$\Psi_{s'[2]02}(\mathbf{u}) = -\frac{1}{8\pi} \int w f_{s'}(\mathbf{u}')\, \frac{d^3\mathbf{u}'}{\gamma'}, \tag{5b}$$

$$\Psi_{s'[3]022}(\mathbf{u}) = -\frac{1}{32\pi} \int c^3 \Big(r \sinh^{-1}(w/c) - w/c\Big) f_{s'}(\mathbf{u}')\, \frac{d^3\mathbf{u}'}{\gamma'}, \tag{5c}$$

$$\Psi_{s'[1]1}(\mathbf{u}) = -\frac{1}{4\pi} \int r w^{-1} f_{s'}(\mathbf{u}')\, \frac{d^3\mathbf{u}'}{\gamma'}, \tag{5d}$$

$$\Psi_{s'[2]11}(\mathbf{u}) = -\frac{1}{8\pi} \int c \sinh^{-1}(w/c) f_{s'}(\mathbf{u}')\, \frac{d^3\mathbf{u}'}{\gamma'}. \tag{5e}$$

These potentials satisfy the differential equations

$$L_0\Psi_{s'[1]0} = f_{s'}, \tag{6a}$$

$$L_2\Psi_{s'[2]02} = \Psi_{s'[1]0}, \tag{6b}$$

$$L_2\Psi_{s'[3]022} = \Psi_{s'[2]02}, \tag{6c}$$

$$L_1\Psi_{s'[1]1} = f_{s'}, \tag{6d}$$

$$L_1\Psi_{s'[2]11} = \Psi_{s'[1]1}, \tag{6e}$$

where

$$L_a\Psi = \left(\mathsf{I} + \frac{\mathbf{uu}}{c^2}\right) : \frac{\partial^2\Psi}{\partial\mathbf{u}\partial\mathbf{u}} + \frac{3\mathbf{u}}{c^2}\cdot\frac{\partial\Psi}{\partial\mathbf{u}} + \frac{1-a^2}{c^2}\Psi. \tag{7}$$

In terms of these potentials the diffusion and friction coefficients are given by

$$\begin{aligned}\mathsf{D}^{s/s'}(\mathbf{u}) = -\frac{4\pi\Gamma^{s/s'}}{n_{s'}}\Bigg[\frac{1}{\gamma}\left(\mathsf{L} + \frac{\mathsf{I}}{c^2} + \frac{\mathbf{uu}}{c^4}\right)\Psi_{s'[2]02} \\ - \frac{4}{\gamma c^2}\left(\mathsf{L} - \frac{\mathsf{I}}{c^2} - \frac{\mathbf{uu}}{c^4}\right)\Psi_{s'[3]022}\Bigg],\end{aligned} \tag{8a}$$

$$\mathbf{F}^{s/s'}(\mathbf{u}) = -\frac{4\pi\Gamma^{s/s'}}{n_{s'}}\frac{m_s}{m_{s'}}\frac{1}{\gamma}\left[\mathbf{K}\Psi_{s'[1]1} - \frac{2}{c^2}\mathbf{K}\Psi_{s'[2]11}\right], \tag{8b}$$

where

$$\mathsf{L}\Psi(\mathbf{u}) = \left(\mathsf{I} + \frac{\mathbf{uu}}{c^2}\right)\cdot\frac{\partial^2\Psi}{\partial\mathbf{u}\partial\mathbf{u}}\cdot\left(\mathsf{I} + \frac{\mathbf{uu}}{c^2}\right) + \left(\mathsf{I} + \frac{\mathbf{uu}}{c^2}\right)\left(\mathbf{u}\cdot\frac{\partial\Psi}{\partial\mathbf{u}}\right),$$

$$\mathbf{K}\Psi(\mathbf{u}) = \left(\mathsf{I} + \frac{\mathbf{uu}}{c^2}\right)\cdot\frac{\partial\Psi}{\partial\mathbf{u}}.$$

Equations (6) and (8) constitute the differential formulation of the relativistic collision operator. Boundary conditions on the solutions of Eqs. (6) are obtained by expansion of Eqs. (5) for $|\mathbf{u}| \to \infty$.

The present notation differs from that in Ref. 5. The correspondence is $\Psi_{s'[1]0} \equiv h_0$, $\Psi_{s'[2]02} \equiv h_1$, $\Psi_{s'[3]022} \equiv h_2$, $\Psi_{s'[1]1} \equiv g_0$, $\Psi_{s'[2]11} \equiv g_1$. The present notation reflects more clearly the structure of the differential equations (6). By adjoining to Eqs. (5) the definition

$$\Psi_{s'[0]}(\mathbf{u}) = f_{s'}(\mathbf{u}),$$

and by using "$*$" in the context $\Psi_{s'[k]*}$ to stand for a string of $k \geq 0$ indices, we can express the five potential equations (6) in the concise form

$$L_a \Psi_{s'[k+1]*a} = \Psi_{s'[k]*}. \tag{9}$$

The integral representations (5) will likewise condense into a single formula

$$\Psi_{s'[k]*}(\mathbf{u}) = \frac{1}{4\pi} \int c^{2k-3} y_{0[k]*}(w/c) f_{s'}(\mathbf{u}') \frac{d^3\mathbf{u}'}{\gamma'}, \tag{10}$$

for $k > 0$. The kernel function $y_{0[k]*}$ is defined in the next section.

III. Solution to potential equations

In this section, Eq. (9) will be solved by separation of variables in spherical coordinates. The choice of spherical coordinates is a natural one since frequently the distribution function is nearly spherically symmetric and so is well represented by only a few spherical harmonics. Since $\Psi_{s[k]*}$ depends on the distribution of species s only, we will simplify the notation by dropping the species subscripts.

A. Spherical harmonic expansion

In a spherical (u, θ, ϕ) coordinate system, the operator L_a is

$$L_a \Psi = \gamma^2 \frac{\partial^2 \Psi}{\partial u^2} + \left(\frac{2}{u} + \frac{3u}{c^2} \right) \frac{\partial \Psi}{\partial u} + \frac{1}{u^2} \left(\frac{\partial^2 \Psi}{\partial \theta^2} + \cot \theta \frac{\partial \Psi}{\partial \theta} \right) + \frac{1}{u^2 \sin^2 \theta} \frac{\partial^2 \Psi}{\partial \phi^2} + \frac{1 - a^2}{c^2} \Psi. \tag{11}$$

Let us expand the potentials in terms of Legendre harmonics:

$$\Psi_{[k]*}(\mathbf{u}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \psi_{lm[k]*}(u) P_l^m(\cos\theta) \exp(im\phi). \tag{12}$$

The coefficients $\psi_{lm[k]*}(u)$ are given by

$$\psi_{lm[k]*}(u) = (2l+1) \frac{(l-m)!}{(l+m)!} \int \frac{d\Omega}{4\pi} \Psi_{[k]*}(\mathbf{u}) P_l^m(\cos\theta) \exp(-im\phi),$$

where $d\Omega$ is an element of solid angle

$$\int d\Omega \ldots = \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi \ldots .$$

Equation (9) becomes

$$L_{l,a}\psi_{lm[k+1]*a} = \psi_{lm[k]*}, \tag{13}$$

where

$$L_{l,a}\chi(u) = \left(1+\frac{u^2}{c^2}\right)\frac{d^2\chi}{du^2}+\left(\frac{2}{u}+\frac{3u}{c^2}\right)\frac{d\chi}{du}-\left(\frac{l(l+1)}{u^2}+\frac{a^2-1}{c^2}\right)\chi, \tag{14}$$

and where $\psi_{lm[0]}(u) = f_{lm}(u)$ is the (l,m) coefficient in the expansion of $f(\mathbf{u})$ in Legendre harmonics.

As before, Eq. (13) stands for a set of five differential equations for the potentials $\psi_{lm[1]0}$, $\psi_{lm[2]02}$, $\psi_{lm[3]022}$, $\psi_{lm[1]1}$, and $\psi_{lm[2]11}$. We will, however, find it convenient to solve for the potentials with arbitrary indices, i.e., to solve the system of equations

$$L_{l,a}\psi_{lm[1]a} = f_{lm}, \tag{15a}$$

$$L_{l,a'}\psi_{lm[2]aa'} = \psi_{lm[1]a}, \tag{15b}$$

$$L_{l,a''}\psi_{lm[3]aa'a''} = \psi_{lm[2]aa'}, \tag{15c}$$

for arbitrary a, a', and a''.

For later reference we list the components of the operators L and $\mathbf{K}$ that occur in Eqs. (8):

$$\mathsf{L}_{uu}\Psi = \gamma^4\,\frac{\partial^2\Psi}{\partial u^2}+\frac{\gamma^2 u}{c^2}\,\frac{\partial\Psi}{\partial u}\,, \tag{16a}$$

$$\mathsf{L}_{\theta\theta}\Psi = \frac{1}{u^2}\,\frac{\partial^2\Psi}{\partial\theta^2}+\frac{\gamma^2}{u}\,\frac{\partial\Psi}{\partial u}\,, \tag{16b}$$

$$\mathsf{L}_{\phi\phi}\Psi = \frac{1}{u^2\sin^2\theta}\,\frac{\partial^2\Psi}{\partial\phi^2}+\frac{\gamma^2}{u}\,\frac{\partial\Psi}{\partial u}+\frac{1}{u^2}\cot\theta\,\frac{\partial\Psi}{\partial\theta}\,, \tag{16c}$$

$$\mathsf{L}_{u\theta}\Psi = \mathsf{L}_{\theta u}\Psi = \frac{\gamma^2}{u}\left(\frac{\partial^2\Psi}{\partial u\partial\theta}-\frac{1}{u}\,\frac{\partial\Psi}{\partial\theta}\right), \tag{16d}$$

$$\mathsf{L}_{u\phi}\Psi = \mathsf{L}_{\phi u}\Psi = \frac{\gamma^2}{u\sin\theta}\left(\frac{\partial^2\Psi}{\partial u\partial\phi}-\frac{1}{u}\,\frac{\partial\Psi}{\partial\phi}\right), \tag{16e}$$

$$\mathsf{L}_{\theta\phi}\Psi = \mathsf{L}_{\phi\theta}\Psi = \frac{1}{u^2\sin\theta}\left(\frac{\partial^2\Psi}{\partial\theta\partial\phi}-\cot\theta\,\frac{\partial\Psi}{\partial\phi}\right), \tag{16f}$$

$$\mathbf{K}_u \Psi = \gamma^2 \frac{\partial \Psi}{\partial u} \,, \tag{16g}$$

$$\mathbf{K}_\theta \Psi = \frac{1}{u} \frac{\partial \Psi}{\partial \theta} \,, \tag{16h}$$

$$\mathbf{K}_\phi \Psi = \frac{1}{u \sin\theta} \frac{\partial \Psi}{\partial \phi} \,, \tag{16i}$$

$$\left(\mathbf{I} + \frac{\mathbf{uu}}{c^2} \right) \Psi = \left(\begin{array}{ccc} \gamma^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \Psi. \tag{16j}$$

B. Homogeneous solutions

In order to solve the inhomogeneous equations (15), it is required first to determine the solutions to the homogeneous equations

$$L_{l,a} \psi^{\mathrm{HS}}_{lm[1]a} = 0, \tag{17a}$$

$$L_{l,a'} \psi^{\mathrm{HS}}_{lm[2]aa'} = \psi^{\mathrm{HS}}_{lm[1]a}, \tag{17b}$$

$$L_{l,a''} \psi^{\mathrm{HS}}_{lm[3]aa'a''} = \psi^{\mathrm{HS}}_{lm[2]aa'}. \tag{17c}$$

It is shown in the appendix that two independent solutions to Eq. (17a) are $\psi^{\mathrm{HS}}_{lm[1]a}(u) = j_{l[1]a}(u/c)$ and $\psi^{\mathrm{HS}}_{lm[1]a}(u) = y_{l[1]a}(u/c)$ where

$$j_{l[1]a}(u/c) = \sqrt{\frac{\pi c}{2u}} P^{-l-1/2}_{a-1/2}(\gamma), \tag{18a}$$

$$y_{l[1]a}(u/c) = (-1)^{-l-1} \sqrt{\frac{\pi c}{2u}} P^{l+1/2}_{a-1/2}(\gamma), \tag{18b}$$

and P^μ_ν is the associated Legendre function of the first kind.

When the full system of equations (17) is considered, we need to introduce the additional functions, defined recursively by Eq. (A6),

$$j_{l[k+2]*aa'} = \begin{cases} \dfrac{j_{l[k+1]*a} - j_{l[k+1]*a'}}{a^2 - a'^2}, & \text{for } a \neq a', \\ \dfrac{\partial j_{l[k+1]*a}}{\partial (a^2)}\,, & \text{for } a = a', \end{cases} \tag{19}$$

with $y_{l[k]*}$ defined in a similar fashion. The general solution to Eqs. (17) is given by

$$\begin{aligned}
\begin{bmatrix} \psi^{\mathrm{HS}}_{lm[1]a} \\ \psi^{\mathrm{HS}}_{lm[2]aa'} \\ \psi^{\mathrm{HS}}_{lm[3]aa'a''} \end{bmatrix} &= C_{lm[1]a} \begin{bmatrix} j_{l[1]a} \\ c^2 j_{l[2]aa'} \\ c^4 j_{l[3]aa'a''} \end{bmatrix} + C'_{lm[1]a} \begin{bmatrix} y_{l[1]a} \\ c^2 y_{l[2]aa'} \\ c^4 y_{l[3]aa'a''} \end{bmatrix} \\
&\quad + C_{lm[2]aa'} \begin{bmatrix} 0 \\ j_{l[1]a'} \\ c^2 j_{l[2]a'a''} \end{bmatrix} + C'_{lm[2]aa'} \begin{bmatrix} 0 \\ y_{l[1]a'} \\ c^2 y_{l[2]a'a''} \end{bmatrix} \\
&\quad + C_{lm[3]aa'a''} \begin{bmatrix} 0 \\ 0 \\ j_{l[1]a''} \end{bmatrix} + C'_{lm[3]aa'a''} \begin{bmatrix} 0 \\ 0 \\ y_{l[1]a''} \end{bmatrix}, \qquad (20)
\end{aligned}$$

where $C_{lm[k]*}$ and $C'_{lm[k]*}$ are arbitrary constants (independent of u) and the argument of the functions $j_{l[k]*}$ and $y_{l[k]*}$ is u/c.

The functions $j_{l[k]*}$ and $y_{l[k]*}$ are invariant under permutation of the indices in $*$ and invariant under change of sign of any index in $*$. Also the functions satisfy $j_{l[k]*}(-z) = (-1)^l j_{l[k]*}(z)$ and $y_{l[k]*}(-z) = (-1)^l y_{l[k]*}(z)$; and $j_{l[k]*}$ and $y_{l[k]*}$ are related by

$$y_{l[k]*} = (-1)^{l+1} j_{-l-1[k]*}.$$

It follows from Eqs. (A11) that $c^{2k+l-2} j_{l[k]*}(u/c)$ and $c^{2k-l-3} y_{l[k]*}(u/c)$ reduce to finite and nonzero expressions in the nonrelativistic limit:

$$\lim_{c\to\infty} c^{2k+l-2} j_{l[k]*}(u/c) = \frac{u^{l+2k-2}}{(2k-2)!!(2l+2k-1)!!}, \qquad (21a)$$

$$\lim_{c\to\infty} c^{2k-l-3} y_{l[k]*}(u/c) = \frac{(-1)^k (2l-2k+1)!!}{(2k-2)!! u^{l-2k+3}}. \qquad (21b)$$

Further properties of these functions are given in the appendix.

For our problem, l and the indices $*$ are always integers, in which case $j_{l[k]*}$ and $y_{l[k]*}$ may be expressed in terms of elementary functions. The functions that we need are given explicitly in Eqs. (A27) and (A28). It is seen there that the kernels appearing in Eqs. (5) are precisely the functions $y_{0[k]*}$; this justifies the general definition for the potentials given in Eq. (10).

C. Green's function

With the solutions of the homogeneous equations in hand, it is a straightforward matter to construct a Green's function and so to write down the general solution of

the inhomogeneous problem. The final task will be to apply appropriate boundary conditions from Eq. (10).

We begin by defining the functions

$$N_{l[0]}(u,u') = 0, \tag{22a}$$

$$N_{l[1]a}(u,u') = c^{-1} y_{l[1]a}(u/c) j_{l[1]a}(u'/c), \tag{22b}$$

$$N_{l[2]aa'}(u,u') = c\Big(y_{l[1]a}(u/c) j_{l[2]aa'}(u'/c) + y_{l[2]aa'}(u/c) j_{l[1]a'}(u'/c)\Big), \tag{22c}$$

$$\begin{aligned} N_{l[3]aa'a''}(u,u') = c^3\Big(&y_{l[1]a}(u/c) j_{l[3]aa'a''}(u'/c) + y_{l[2]aa'}(u/c) j_{l[2]a'a''}(u'/c) \\ &+ y_{l[3]aa'a''}(u/c) j_{l[1]a''}(u'/c)\Big). \end{aligned} \tag{22d}$$

From Eqs. (21), we see that the functions $N_{l[k]*}$ reduce to finite nonzero expressions in the nonrelativistic limit. These functions satisfy the differential equations

$$L_{l,a} N_{l[k+1]*a}(u,u') = N_{l[k]*}(u,u'),$$
$$L_{l,a} N_{l[k+1]*a}(u',u) = N_{l[k]*}(u',u).$$

Next we combine $N_{l[k]*}(u,u')$ and $N_{l[k]*}(u',u)$ to obtain the Green's functions

$$K_{l[k]*}(u,u') = \begin{cases} \dfrac{\gamma}{u^2}\delta(u-u'), & \text{for } k=0 \\ N_{l[k]*}(u_>,u_<), & \text{for } k>0, \end{cases} \tag{23}$$

where $u_> = \max(u,u')$ and $u_< = \min(u,u')$. These Green's functions satisfy

$$L_{l,a} K_{l[k+1]*a}(u,u') = K_{l[k]*}(u,u').$$

To establish this relation, we use

$$\frac{\partial}{\partial u}\left(N_{l[k]*}(u,u') - N_{l[k]*}(u',u)\right)\Big|_{u=u'} = \begin{cases} \dfrac{1}{\gamma u^2}, & \text{for } k=1, \\ 0, & \text{otherwise,} \end{cases}$$

where the result for $k = 1$ follows from the expression for the Wronskian, Eq. (A5). Furthermore, the Green's functions satisfy equations analogous to Eqs. (19), namely,

$$\frac{K_{l[k+2]*aa'}}{c^2} = \begin{cases} \dfrac{K_{l[k+1]*a} - K_{l[k+1]*a'}}{a^2 - a'^2}, & \text{for } a \neq a', \\ \dfrac{\partial K_{l[k+1]*a}}{\partial(a^2)}, & \text{for } a = a'. \end{cases} \tag{24}$$

The Green's functions $K_{l[k]*}$ are therefore symmetric under interchange of the indices in $*$.

A particular integral for the differential equations (13) is now given by

$$\psi^{\rm PI}_{lm[k]*}(u) = \int_0^\infty K_{l[k]*}(u,u')\frac{u'^2}{\gamma'}f_{lm}(u')\,du'. \tag{25}$$

We shall show next that this particular integral is precisely the solution defined by the integral form of the potential Eq. (10). This will be done by matching the behavior of Eq. (25) near $u=0$ to the behavior of the solutions of Eq. (10). For $u\to 0$ we have $K_{l[k]*}(u,u')\to N_{l[k]*}(u',u)$. By using Eqs. (22) and (A11a) we obtain in this limit

$$\psi^{\rm PI}_{lm[k]*}(u) \to \frac{u^l c^{2k-l-3}}{(2l+1)!!}\int_0^\infty y_{l[k]*}(u'/c)\frac{u'^2}{\gamma'}f_{lm}(u')\,du'. \tag{26}$$

The general solution is obtained by adding the homogeneous solution, Eq. (20), to the particular integral: $\psi_{lm[k]*} = \psi^{\rm PI}_{lm[k]*} + \psi^{\rm HS}_{lm[k]*}$. It remains to determine the coefficients $C_{lm[k]*}$ and $C'_{lm[k]*}$ appearing in Eq. (20). The primed coefficients $C'_{lm[k]*}$ are found by considering just the order of growth of $\psi_{lm[k]*}(u)$ at $u=0$. From Eq. (12) it follows that $\psi_{lm[k]*}(u) = O(u^l)$, whereas from Eqs. (20) and (A11), we have $\psi^{\rm HS}_{lm[k]*} = C'_{lm[k]*}O(u^{-l-1})$. The coefficients $C'_{lm[k]*}$ must vanish in order to suppress this divergence at $u=0$. To establish that the unprimed coefficients $C_{lm[k]*}$ also vanish, we must consider more carefully the behavior of $\psi_{lm[k]*}(u)$ at the origin.

From Eqs. (20), (26), and (A11a), we find that the leading order behavior of $\psi_{lm[k]*}$ is

$$\psi_{lm[k]*} \to \left[C_{lm[k]*} + c^{2k-3}\int_0^\infty y_{l[k]*}(u'/c)\frac{u'^2}{\gamma'}f_{lm}(u')\,du'\right]\frac{(u/c)^l}{(2l+1)!!}. \tag{27}$$

In order to determine $C_{lm[k]*}$, we expand the integral representation of $\psi_{lm[k]*}$ near $u=0$. Substituting the spherical harmonic expansion, Eq. (12), into Eq. (10), we obtain

$$\psi_{lm[k]*}(u) = c^{2k-3}\int_0^\infty u'^2\,du'\int\frac{d\Omega}{4\pi}\,y_{0[k]*}(w/c)\frac{f_{lm}(u')}{\gamma'}\frac{Y_{lm}(\theta',\phi')}{Y_{lm}(\theta,\phi)}, \tag{28}$$

where we have introduced the spherical harmonics[6]

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)\exp(im\phi).$$

Because we want to expand Eq. (28) for small u, we write w from Eq. (4) as

$$w = \sqrt{u'^2 \left(1 + \frac{\epsilon^2}{c^2}\right) - 2\epsilon u' \gamma'},$$

where

$$\epsilon = u \cos\alpha - c^2(\gamma - 1)\frac{\gamma'}{u'},$$

and $\alpha = \cos^{-1}(\mathbf{u} \cdot \mathbf{u}'/uu')$ is the angle between $\mathbf{u}$ and $\mathbf{u}'$. Because the u dependence only enters through ϵ and because $\epsilon = O(u)$, we proceed by expanding Eq. (28) for small ϵ. We use Eq. (A24) to expand $y_{0[k]*}(w/c)$ as

$$y_{0[k]*}(w/c) = \sum_{l'=0}^{\infty} c^{-l'} y_{l'[k]*}(u'/c)\frac{\epsilon^{l'}}{l'!}. \tag{29}$$

When we substitute for ϵ and perform the angle integrations, we encounter integrals of the form

$$I_{nl} = \int \frac{d\Omega}{4\pi} \cos^n\alpha \, Y_{lm}(\theta', \phi').$$

Using formula (7.126.1) of Gradshteyn and Ryzhik,[7] we can expand $\cos^n\alpha$ as

$$\cos^n\alpha = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^k k!} \frac{2n - 4k + 1}{(2n - 2k + 1)!!} P_{n-2k}(\cos\alpha).$$

Likewise, we can use Eq. (3.62) of Jackson[6] to expand $P_n(\cos\alpha)$ as

$$P_n(\cos\alpha) = \frac{4\pi}{2n+1} \sum_{k=-n}^{n} Y^*_{nk}(\theta', \phi') Y_{nk}(\theta, \phi).$$

Substituting these series into I_{nl} and using the orthogonality condition for the spherical harmonics

$$\int d\Omega \, Y^*_{l'm'}(\theta, \phi) Y_{lm}(\theta, \phi) = \delta_{l'l}\delta_{m'm},$$

we obtain

$$I_{nl} = \begin{cases} \dfrac{n!}{2^{(n-l)/2}((n-l)/2)!(n+l+1)!!}, & \text{for } n \geq l \text{ and } n - l \text{ even,} \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, the first term in the sum in Eq. (29) which contributes to the integral in Eq. (28) is the term $l' = l$. This results in a term of order u^l. The remaining terms in the sum contribute terms of higher order in u. Thus, we have

$$\psi_{lm[k]*}(u) \to \frac{u^l c^{2k-l-3}}{(2l+1)!!} \int_0^\infty y_{l[k]*}(u'/c) \frac{u'^2}{\gamma'} f_{lm}(u')\, du'. \tag{30}$$

Comparing Eqs. (27) and (30), we find that $C_{lm[k]*} = 0$, and therefore that $\psi^{\mathrm{HS}}_{lm[k]*} = 0$. The desired solution is given just by the particular integral, Eq. (25), which for $k > 0$ we may write as

$$\begin{aligned}\psi_{lm[k]*}(u) = \int_0^u N_{l[k]*}(u, u') \frac{u'^2}{\gamma'} f_{lm}(u')\, du' \\ + \int_u^\infty N_{l[k]*}(u', u) \frac{u'^2}{\gamma'} f_{lm}(u')\, du',\end{aligned} \tag{31}$$

with $N_{l[k]*}$ given by Eqs. (22). This completes the solution for the potentials $\psi_{lm[k]*}$.

In obtaining this result, we have, in effect, found a spherical harmonic decomposition of the kernel in Eq. (10)

$$\begin{aligned}c^{2k-3} y_{0[k]*}(w/c) &= 4\pi \sum_{l=0}^\infty \sum_{m=-l}^{l} N_{l[k]*}(u_>, u_<) Y^*_{lm}(\theta', \phi') Y_{lm}(\theta, \phi) \\ &= \sum_{l=0}^\infty (2l+1) N_{l[k]*}(u_>, u_<) P_l(\cos\alpha).\end{aligned}$$

For $k = 1$, this yields an addition formula for a class of associated Legendre functions:

$$\frac{P^{1/2}_{a-1/2}(r)}{(r^2-1)^{1/4}} = \sum_{l=0}^\infty (-1)^l (2l+1) \sqrt{\frac{\pi}{2}}\, \frac{P^{l+1/2}_{a-1/2}(\gamma)}{(\gamma^2-1)^{1/4}}\, \frac{P^{-l-1/2}_{a-1/2}(\gamma')}{(\gamma'^2-1)^{1/4}}\, P_l(\cos\alpha),$$

where $r = \gamma\gamma' - \sqrt{\gamma^2-1}\sqrt{\gamma'^2-1}\cos\alpha$ and $1 < \gamma' \le \gamma$. This identity is not found in the usual handbooks, although it did turn out to be known.[8,9]

IV. Nonrelativistic limit

Taking the limit $c \to \infty$ in the preceding equations, we recover the well-known nonrelativistic collision operator. We catalog here the important results.

The kernel U reduces to the one given by Landau[1]

$$\mathsf{U}(\mathbf{u},\mathbf{u}') = \frac{1}{|\mathbf{u}-\mathbf{u}'|^3}\Big(|\mathbf{u}-\mathbf{u}'|^2\,\mathsf{I} - (\mathbf{u}-\mathbf{u}')(\mathbf{u}-\mathbf{u}')\Big).$$

The quantities r and w reduce to 1 and $|\mathbf{u}-\mathbf{u}'|$, respectively. The potentials $\Psi_{s'[k]*}$ only depend on the number of indices, k, and not on their values. We will, therefore, drop the indices and write $\Psi_{s'[k]}(\mathbf{u})$ instead of $\Psi_{s'[k]*}(\mathbf{u})$. When Eq. (21b) is substituted into Eq. (10), the potentials become

$$\Psi_{s'[k]}(\mathbf{u}) = \begin{cases} f_{s'}(\mathbf{u}), & \text{for } k=0,\\ -\dfrac{1}{4\pi}\displaystyle\int \frac{|\mathbf{u}-\mathbf{u}'|^{2k-3}}{(2k-2)!} f_{s'}(\mathbf{u}')\, d^3\mathbf{u}', & \text{for } k>0. \end{cases}$$

The potentials satisfy $L\Psi_{s'[k+1]} = \Psi_{s'[k]}$, where L is now the velocity-space Laplacian,

$$L\Psi = \frac{\partial}{\partial \mathbf{u}}\cdot\frac{\partial \Psi}{\partial \mathbf{u}}.$$

In particular, we have

$$\begin{aligned} L\Psi_{s'[1]} &= f_{s'},\\ L\Psi_{s'[2]} &= \Psi_{s'[1]}. \end{aligned}$$

The diffusion and friction coefficients are given by

$$\begin{aligned} \mathsf{D}^{s/s'}(\mathbf{u}) &= -\frac{4\pi\Gamma^{s/s'}}{n_{s'}}\,\frac{\partial^2}{\partial\mathbf{u}\partial\mathbf{u}}\Psi_{s'[2]},\\ \mathbf{F}^{s/s'}(\mathbf{u}) &= -\frac{4\pi\Gamma^{s/s'}}{n_{s'}}\,\frac{m_s}{m_{s'}}\,\frac{\partial}{\partial\mathbf{u}}\Psi_{s'[1]}. \end{aligned}$$

The homogeneous solutions to the separated radial components of the nonrelativistic potential equations are given by Eqs. (21). Substituting these into Eqs. (22) and then substituting the result into Eq. (31), we obtain the Legendre harmonic expansion for the potentials

$$\begin{aligned} &\psi_{lm[1]}(u) = \\ &\quad -\int_0^u \frac{1}{2l+1}\frac{u'^l}{u^{l+1}}u'^2 f_{lm}(u')\,du' - \int_u^\infty \frac{1}{2l+1}\frac{u^l}{u'^{l+1}}u'^2 f_{lm}(u')\,du', \end{aligned}$$

$$\psi_{lm[2]}(u) = \\ -\int_0^u \frac{1}{2(2l+1)}\left[\frac{1}{2l+3}\frac{u'^{l+2}}{u^{l+1}} - \frac{1}{2l-1}\frac{u'^{l}}{u^{l-1}}\right] u'^2 f_{lm}(u')\,du' \\ -\int_u^\infty \frac{1}{2(2l+1)}\left[\frac{1}{2l+3}\frac{u^{l+2}}{u'^{l+1}} - \frac{1}{2l-1}\frac{u^{l}}{u'^{l-1}}\right] u'^2 f_{lm}(u')\,du'.$$

These expressions coincide with the results of Rosenbluth, MacDonald, and Judd.[3]

V. Isotropic background

In this section and in Sec. VI, we consider the cases where the background is described by the $l = 0$, $m = 0$ and $l = 1$, $m = 0$ components in Eq. (12). First let us consider the azimuthally symmetric case, $m = 0$ and l arbitrary. For simplicity we will drop the m subscript and thus write

$$\Psi_{[k]*}(\mathbf{u}) = \sum_{l=0}^{\infty} \psi_{l[k]*}(u) P_l(\cos\theta).$$

When we substitute this representation into Eqs. (16), the second derivatives in the component $\mathsf{L}_{uu}\Psi$ may be eliminated by using the differential equation (13); this gives

$$\mathsf{L}_{uu}\Big(\psi_{l[k+1]*a}(u)P_l(\cos\theta)\Big) = \left[\gamma^2\psi_{l[k]*} - \frac{2\gamma^4}{u}\,\frac{\partial\psi_{l[k+1]*a}}{\partial u} \right. \\ \left. + \gamma^2\left(\frac{l(l+1)}{u^2} + \frac{a^2-1}{c^2}\right)\psi_{l[k+1]*a}\right] P_l(\cos\theta). \tag{32}$$

Let us now specialize to an isotropic background $l = 0$, $P_0(\cos\theta) = 1$. If we substitute Eqs. (16) and (32) into Eqs. (8), we obtain

$$D^{s/s'}_{uu,0} = \frac{4\pi\Gamma^{s/s'}}{n_{s'}}\frac{\gamma}{u}\left[2\gamma^2\frac{d\psi_{s'0[2]02}}{du} - u\psi_{s'0[1]0} \right. \\ \left. - \frac{8\gamma^2}{c^2}\frac{d\psi_{s'0[3]022}}{du} + \frac{8u}{c^4}\psi_{s'0[3]022}\right], \tag{33a}$$

$$
\begin{aligned}
D^{s/s'}_{\theta\theta,0} = \frac{4\pi\Gamma^{s/s'}}{n_{s'}} \frac{1}{\gamma u} \biggl[-\gamma^2 \frac{d\psi_{s'0[2]02}}{du} - \frac{u}{c^2}\psi_{s'0[2]02} \\
+ \frac{4\gamma^2}{c^2}\frac{d\psi_{s'0[3]022}}{du} - \frac{4u}{c^4}\psi_{s'0[3]022}\biggr],
\end{aligned} \tag{33b}
$$

$$
F^{s/s'}_{u,0} = \frac{4\pi\Gamma^{s/s'}}{n_{s'}} \frac{m_s}{m_{s'}}\gamma \biggl[-\frac{d\psi_{s'0[1]1}}{du} + \frac{2}{c^2}\frac{d\psi_{s'0[2]11}}{du}\biggr]. \tag{33c}
$$

The other components of $\mathsf{D}^{s/s'}_0$ and $\mathbf{F}^{s/s'}_0$ vanish. Lastly we substitute for the potentials using Eq. (31). Some massaging of the result leads to

$$
\begin{aligned}
D^{s/s'}_{uu,0} = \frac{4\pi\Gamma^{s/s'}}{n_{s'}} \biggl\{ \int_0^u \Bigl[2\gamma^2 c^2 j'_{0[2]02} - 8c^2 j'_{0[3]022}\Bigr] \frac{\gamma}{u^3}\frac{u'^2}{\gamma'} f_{s'0}(u')\,du' \\
+ \int_u^\infty \Bigl[2\gamma'^2 c^2 j_{0[2]02} - 8c^2 j_{0[3]022}\Bigr] \frac{\gamma}{u^2}\frac{u'}{\gamma'} f_{s'0}(u')\,du' \biggr\},
\end{aligned} \tag{34a}
$$

$$
\begin{aligned}
D^{s/s'}_{\theta\theta,0} = \frac{4\pi\Gamma^{s/s'}}{n_{s'}} \biggl\{ \int_0^u \biggl[\frac{1}{2} j'_{0[1]2} - \biggl(\frac{c^2}{u^2} + \frac{1}{\gamma^2}\biggr) j'_{0[2]02} \\
+ \frac{4}{\gamma^2}\frac{c^2}{u^2} j'_{0[3]022}\biggr] \frac{\gamma}{u}\frac{u'^2}{\gamma'} f_{s'0}(u')\,du' \\
+ \int_u^\infty \biggl[\frac{1}{2}\frac{\gamma'^2}{\gamma^2} j_{0[1]2} - \frac{u'^2}{u^2}\biggl(\frac{c^2}{u'^2} + \frac{1}{\gamma^2}\biggr) j_{0[2]02} \\
+ \frac{4}{\gamma^2}\frac{c^2}{u^2} j_{0[3]022}\biggr] \gamma\frac{u'}{\gamma'} f_{s'0}(u')\,du' \biggr\},
\end{aligned} \tag{34b}
$$

$$
\begin{aligned}
F^{s/s'}_{u,0} = -\frac{4\pi\Gamma^{s/s'}}{n_{s'}} \frac{m_s}{m_{s'}} \biggl\{ \int_0^u \Bigl[\gamma^2 j'_{0[1]1} - 2 j'_{0[2]11}\Bigr] \frac{1}{u^2}\frac{u'^2}{\gamma'} f_{s'0}(u')\,du' \\
+ \int_u^\infty 4\frac{u'}{u} j_{0[2]02} f_{s'0}(u')\,du' \biggr\},
\end{aligned} \tag{34c}
$$

where $j_{l[k]*} = j_{l[k]*}(u/c)$ and $j'_{l[k]*} = j_{l[k]*}(u'/c)$.

Of particular interest is the case of a Maxwellian background, i.e.,

$$
f_{s'0}(u) = f_{s'm} = \frac{n_{s'}m_{s'}}{4\pi c T_{s'} K_2(m_{s'}c^2/T_{s'})} \exp\biggl(-\frac{m_{s'}c^2\gamma}{T_{s'}}\biggr),
$$

where K_n is the nth order Bessel function of the second kind. First of all we can verify that

$$F_{u,0}^{s/s'} = -\frac{m_s v}{T_{s'}} D_{uu,0}^{s/s'},$$

where $v = u/\gamma$. This is accomplished by substituting

$$f_{s'm}(u') = -\frac{T_{s'}}{m_{s'} v'} \frac{d}{du'} f_{s'm}(u')$$

into the expression for $D_{uu,0}^{s/s'}$ and integrating by parts. This relation between $F_{u,0}^{s/s'}$ and $D_{uu,0}^{s/s'}$ implies that the collisions will cause f_s to relax to a Maxwellian with temperature $T_{s'}$.

In the high-energy limit $m_{s'} c^2(\gamma - 1) \gg T_{s'}$, the indefinite limits in the integrals in Eqs. (34) can be replaced by ∞. We can perform the resulting integrals using formula (7.141.5) of Gradshteyn and Ryzhik[7] which gives, after a change of integration variable,

$$\int_0^\infty \exp\Bigl(-\beta\sqrt{1+z^2}\Bigr) \frac{z^{l+2}}{\gamma} j_{l[1]a}(z)\, dz = \frac{K_a(\beta)}{\beta^{l+1}}.$$

For $l = 0$ and $a = 2$ this gives the normalization condition for the Maxwellian

$$\int_0^\infty 4\pi u^2 f_{s'm}(u)\, du = n_{s'}.$$

On carrying out the integrations in Eqs. (34), we obtain

$$D_{uu,0}^{s/s'} = \Gamma^{s/s'} \frac{u_{ts'}^2}{v^3} \frac{K_1}{K_2} \left(1 - \frac{K_0}{K_1} \frac{u_{ts'}^2}{\gamma^2 c^2}\right), \tag{35a}$$

$$D_{\theta\theta,0}^{s/s'} = \Gamma^{s/s'} \frac{1}{2v} \left[1 - \frac{K_1}{K_2} \left(\frac{u_{ts'}^2}{u^2} + \frac{u_{ts'}^2}{\gamma^2 c^2}\right) + \frac{K_0}{K_2} \frac{u_{ts'}^2}{u^2} \frac{u_{ts'}^2}{\gamma^2 c^2}\right], \tag{35b}$$

$$F_{u,0}^{s/s'} = -\Gamma^{s/s'} \frac{m_s}{m_{s'}} \frac{1}{v^2} \frac{K_1}{K_2} \left(1 - \frac{K_0}{K_1} \frac{u_{ts'}^2}{\gamma^2 c^2}\right), \tag{35c}$$

where $u_{ts}^2 = T_s/m_s$ and the argument for the Bessel functions is $m_{s'} c^2/T_{s'}$. In the limit $m_{s'} \to \infty$, we recover the Lorentz collision operator

$$D_{uu,0} = F_{u,0} = 0, \tag{36a}$$

$$D_{\theta\theta,0} = \Gamma^{s/s'} \frac{1}{2v}. \tag{36b}$$

VI. First harmonic

In addition to the isotropic components of the collision operator calculated in the previous section, the first harmonic of the Legendre expansion is also required in the calculation of the electrical conductivity. Specifically, we need to compute the term $C^{s/s'}(f_{sm}, f_{s'1}\cos\theta)$. We can express this in terms of the potentials and their derivatives using Eqs. (1), (8), and (16) to give

$$
\begin{aligned}
&\frac{C^{s/s'}(f_{sm}(u), f_{s'1}(u)\cos\theta)}{f_{sm}(u)\cos\theta} \\
&\quad = \frac{4\pi\Gamma^{s/s'}}{n_{s'}}\Bigg\{\frac{m_s}{m_{s'}}\Bigg[\frac{1}{\gamma}\psi_{s'1[0]} - \frac{u}{u_{ts}^2}\frac{d\psi_{s'1[1]1}}{du} \\
&\qquad\qquad - \frac{2}{c^2\gamma}\psi_{s'1[1]1} + \frac{2u}{c^2u_{ts}^2}\frac{d\psi_{s'1[2]11}}{du}\Bigg] \\
&\qquad + \frac{u}{u_{ts}^2}\frac{d\psi_{s'1[1]0}}{du} - \left(\frac{u^2}{\gamma u_{ts}^4} - \frac{1}{u_{ts}^2}\right)\psi_{s'1[1]0} \\
&\qquad + \left(\frac{2\gamma u}{u_{ts}^4} - \frac{2u}{c^2u_{ts}^2}\right)\frac{d\psi_{s'1[2]02}}{du} - \left(\frac{2}{\gamma u_{ts}^4} - \frac{2}{c^2u_{ts}^2}\right)\psi_{s'1[2]02} \\
&\qquad - \frac{8\gamma u}{c^2u_{ts}^4}\frac{d\psi_{s'1[3]022}}{du} + \frac{8\gamma}{c^2u_{ts}^4}\psi_{s'1[3]022}\Bigg\}.
\end{aligned}
\tag{37}
$$

Finally, we substitute for the potentials and simplify to obtain

$$
\begin{aligned}
&\frac{C^{s/s'}(f_{sm}(u), f_{s'1}(u)\cos\theta)}{f_{sm}(u)\cos\theta} \\
&\quad = \frac{4\pi\Gamma^{s/s'}}{n_{s'}}\Bigg\{\frac{m_s}{m_{s'}}\frac{1}{\gamma}f_{s'1}(u) \\
&\qquad + \int_0^u\Bigg[\frac{1}{u^2}\left(2\frac{m_s}{m_{s'}}\frac{j'_{1[1]1}}{c^2} + \frac{j'_{1[1]2}}{u_{ts}^2} - 10\frac{j'_{1[2]02}}{u_{ts}^2}\right) \\
&\qquad\quad + \frac{\gamma}{u^2}\Bigg(-2\frac{m_s}{m_{s'}}\frac{j'_{1[1]1}}{u_{ts}^2} + 4\frac{m_s}{m_{s'}}\frac{j'_{1[2]11}}{u_{ts}^2} \\
&\qquad\qquad + 6\frac{c^2j'_{1[2]02}}{u_{ts}^4} - 24\frac{c^2j'_{1[3]022}}{u_{ts}^4}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{j'_{1[1]0}}{c^2u_{ts}^2}\right)+\gamma\left(2\frac{j'_{1[2]02}}{u_{ts}^4}\right)\Bigg]\frac{cu'^2}{\gamma\gamma'}f_{s'1}(u')\,du' \\
+\int_u^\infty\Bigg[&\frac{1}{u'^2}\left(2\frac{m_s}{m_{s'}}\frac{j_{1[1]1}}{c^2}+\frac{m_s}{m_{s'}}\frac{j_{1[1]2}}{u_{ts}^2}-10\frac{m_s}{m_{s'}}\frac{j_{1[2]02}}{u_{ts}^2}\right) \\
&+\frac{\gamma'}{u'^2}\Bigg(-2\frac{j_{1[1]1}}{u_{ts}^2}+4\frac{j_{1[2]11}}{u_{ts}^2} \\
&\qquad\qquad +6\frac{c^2j_{1[2]02}}{u_{ts}^4}-24\frac{c^2j_{1[3]022}}{u_{ts}^4}\Bigg) \\
&+\left(\frac{m_s}{m_{s'}}\frac{j_{1[1]0}}{c^2u_{ts}^2}\right)+\gamma'\left(2\frac{j_{1[2]02}}{u_{ts}^4}\right)\Bigg]\frac{cu'^2}{\gamma\gamma'}f_{s'1}(u')\,du'\Bigg\}, \qquad (38)
\end{aligned}
$$

where, as before, $j_{l[k]*}=j_{l[k]*}(u/c)$ and $j'_{l[k]*}=j_{l[k]*}(u'/c)$.

Both in Eqs. (34) and in Eq. (38), one can substitute for $j_{l[k]*}$ from Eqs. (A27) and (A28) and thereby express the collision operator entirely in terms of elementary functions. The resulting expressions would be very badly behaved numerically near $u=0$ because of large cancellations. It is preferable, therefore, to evaluate $j_{l[k]*}$ directly by the method outlined in the appendix.

The collision operator obeys the conservation law

$$\int\left[h_sC^{s/s'}(f_s,f_{s'})+h_{s'}C^{s'/s}(f_{s'},f_s)\right]d^3\mathbf{u}=0,$$

where $h_s=a_0+\mathbf{a}_1\cdot m_s\mathbf{u}+a_2m_sc^2\gamma$, and a_0, $\mathbf{a}_1$, and a_2 are arbitrary constants. The collision operator is also self-adjoint:

$$\int\psi C^{s/s'}(\chi f_{sm},f_{s'm})\,d^3\mathbf{u}=\int\chi C^{s/s'}(\psi f_{sm},f_{s'm})\,d^3\mathbf{u},$$

and satisfies the symmetry

$$\int\psi C^{s/s'}(f_{sm},\chi f_{s'm})\,d^3\mathbf{u}=\int\chi C^{s'/s}(f_{s'm},\psi f_{sm})\,d^3\mathbf{u},$$

where ψ and χ are arbitrary functions of $\mathbf{u}$, and $T_s=T_{s'}$. Combining these two properties gives

$$C^{s/s'}(h_sf_{sm},f_{s'm})+C^{s/s'}(f_{sm},h_{s'}f_{s'm})=0.$$

This provides a useful check on the implementations of Eqs. (34) and (38)

VII. Calculation of the conductivity

At this point the calculation of the electrical conductivity is straightforward. We consider an electron-ion plasma with infinitely massive stationary ions; $m_i \to \infty$ and $f_{im} \to n_i \delta(\mathbf{u})$. In the presence of a weak electric field $E\hat{\mathbf{z}}$, the electron distribution is given to first order by $f_{em}(1 + \chi_1(u,t)\cos\theta)$. The linearized Boltzmann equation may be written in the form

$$\frac{\partial \chi_1}{\partial t} = \frac{q_e E v}{T_e} + \hat{C}_e(\chi_1), \tag{39}$$

where $\hat{C}_e(\chi_1)$ is the linearized electron collision term

$$\hat{C}_e(\chi_1) = \frac{1}{f_{em}\cos\theta}\Big[C^{e/e}(f_{em}\chi_1\cos\theta, f_{em}) + C^{e/e}(f_{em}, f_{em}\chi_1\cos\theta) \\ + C^{e/i}(f_{em}\chi_1\cos\theta, f_{im})\Big].$$

The first term here is given by

$$\frac{C^{e/e}(f_{em}\chi_1\cos\theta, f_{em})}{f_{em}\cos\theta} = \frac{1}{u^2}\frac{\partial}{\partial u}u^2 D^{e/e}_{uu,0}\frac{\partial \chi_1}{\partial u} + F^{e/e}_{u,0}\frac{\partial \chi_1}{\partial u} - \frac{2}{u^2}D^{e/e}_{\theta\theta,0}\chi_1,$$

with D and $\mathbf{F}$ given by Eqs. (34). The second is given directly by Eq. (38) with $s = s' = e$ and $f_{s'1} = f_{s'm}\chi_1$. The last term is given by the Lorentz limit, Eqs. (36),

$$\frac{C^{e/i}(f_{em}\chi_1\cos\theta, f_{im})}{f_{em}\cos\theta} = -\frac{\Gamma^{e/i}}{u^2 v}\chi_1.$$

The conductivity is defined by

$$\sigma = \frac{4\pi q_e}{3E}\int_0^\infty f_{em}(u)\chi_1(u, t\to\infty) v u^2\, du.$$

The time asymptotic solution $\chi_1(u, t \to \infty)$ is determined by solving Eq. (39) as an initial value problem. We take $\chi_1(u, t = 0) = 0$ (for example). The differential terms $C^{e/e}(f_{em}\chi_1\cos\theta, f_{em})$ and $C^{e/i}(f_{em}\chi_1\cos\theta, f_{im})$ are both treated fully implicitly, while the integral term $C^{e/e}(f_{em}, f_{em}\chi_1\cos\theta)$ is treated explicitly. This permits large time steps to be taken and leads to a rapid convergence to a steady state. A check on χ_1 is obtained by evaluating the first moment of the linearized Boltzmann equation. The electron-electron collision terms drop out by conservation of momentum, to give

$$\frac{4\pi}{3}\int_0^\infty f_{em}\chi_1\gamma\, du = \frac{n_e q_e E}{m_e Z \Gamma^{e/e}}, \tag{40}$$

where

$$Z = \frac{\Gamma^{e/i}}{\Gamma^{e/e}} = -\frac{q_i \log \Lambda^{e/i}}{q_e \log \Lambda^{e/e}}$$

is the effective ion charge state, and where we have assumed $q_e n_e + q_i n_i = 0$.

It is convenient to write σ as

$$\begin{aligned}\sigma &= \frac{n_e q_e^2 u_{te}^3}{m_e Z \Gamma^{e/e}} \bar{\sigma}(\Theta, Z) \\ &= \frac{4\pi\epsilon_0^2}{m_e^{1/2} q_e^2 \log \Lambda^{e/e}} \frac{T_e^{3/2}}{Z} \bar{\sigma}(\Theta, Z),\end{aligned}$$

where

$$\Theta = \frac{T_e}{m_e c^2} = \frac{T_e}{511\,\mathrm{keV}}, \qquad u_{te} = \sqrt{T_e/m_e}.$$

The normalized conductivity $\bar{\sigma}$ is a dimensionless function of two dimensionless arguments. In the limit $\Theta \to 0$, $\bar{\sigma}$ is bounded and nonzero, and we recover the nonrelativistic scaling $\sigma \propto T_e^{3/2}$. Values of $\bar{\sigma}$ for various Θ and Z are tabulated in Table 1 and plotted in Fig. 1. The nonrelativistic conductivity was first calculated by Spitzer and Härm,[10] who quote values of $\bar{\sigma}(0, Z)/\bar{\sigma}(0, \infty)$. Their results coincide with ours in the limit $\Theta \to 0$.

In the limit $Z \to \infty$, electron-electron collisions can be ignored, and the relevant collision term is the Lorentz electron-ion collision term, Eqs. (36). This case is considered by Lifshitz and Pitaevskii.[11] We can write

$$\chi_1 = \frac{q_e E}{Z T_e \Gamma^{e/e}} u^2 v^2.$$

The resulting conductivity is

$$\begin{aligned}\bar{\sigma} &= \frac{4\pi}{3} \frac{1}{n_e u_{te}^5} \int_0^\infty u^4 v^3 f_{em}\, du \\ &= \frac{1}{3\Theta^{7/2} K_2(\Theta^{-1})} \int_1^\infty \frac{(\gamma^2 - 1)^3}{\gamma^2} \exp\left(-\frac{\gamma}{\Theta}\right) d\gamma.\end{aligned}$$

Evaluating the integral, we obtain

$$\begin{aligned}\bar{\sigma} = \frac{1}{3\Theta^{7/2} K_2(\Theta^{-1})} \Bigg(&\frac{E_1(\Theta^{-1})}{\Theta} \\ &- (1 - \Theta + 2\Theta^2 - 6\Theta^3 - 24\Theta^4 - 24\Theta^5) \exp(\Theta^{-1}) \Bigg), \end{aligned} \tag{41}$$

where E_n is the exponential integral. In the limit $\Theta \to 0$, this reduces to $\bar{\sigma} = 16\sqrt{2/\pi}$. For $\Theta \to \infty$, we obtain $\bar{\sigma} = 4/\sqrt{\Theta}$, which agrees with the result of Lifshitz and Pitaevskii.

Another tractable, albeit less interesting, limit is $Z \to 0$. In this case, this electrons equilibrate with themselves so that their distribution is a Maxwellian drifting at v_d and $\chi_1 = v_d u/u_{te}^2$. The drift speed v_d is found by applying Eq. (40) to give

$$v_d = \frac{3\exp(\Theta^{-1})K_2(\Theta^{-1})}{\sqrt{\Theta}(1+2\Theta+2\Theta^2)} \frac{q_e E u_{te}^3}{m_e Z \Gamma^{e/e}}.$$

The resulting conductivity is

$$\bar{\sigma} = \frac{3\exp(\Theta^{-1})K_2(\Theta^{-1})}{\sqrt{\Theta}(1+2\Theta+2\Theta^2)}. \tag{42}$$

In this expression we recognize the result obtained by van Erkelens and van Leeuwen[12] on the basis of a lowest-order variational treatment of the relativistic Boltzmann equation. Their result for the conductivity of a relativistic plasma, therefore, corresponds to the limit $Z \to 0$. For this case the limit $\Theta \to 0$ gives $\bar{\sigma} = 3\sqrt{\pi/2}$ and the limit $\Theta \to \infty$ gives $\bar{\sigma} = 3/\sqrt{\Theta}$.

VIII. Conclusions

In our earlier work,[5] we gave a differential formulation for the collision operator for a relativistic plasma. This formulation is summarized by Eqs. (6) and (8). A major objective of the present work is to solve the differential equations (6) and, hence, to express the potentials in terms of quadrature. This was achieved by using an expansion in Legendre harmonics, Eq. (12). The radial components of the potentials are then given by Eq. (31) where the kernels $N_{l[k]*}(u, u')$ are given by Eqs. (22); these in turn involve the special functions $j_{l[k]*}$ and $y_{l[k]*}$, whose properties are given in the appendix. The entire formulation is well-behaved in the nonrelativistic limit; indeed, in this limit, the potentials and their solutions agree with the earlier nonrelativistic treatment of Rosenbluth *et al.*[3] and Trubnikov.[4]

Several computer codes exist which solve the nonlinear Fokker-Planck equation in the nonrelativistic limit. In many of these codes the collision operator is evaluated in terms of a Legendre harmonic expansion of the potentials. Our results are easily incorporated into such codes, allowing them to treat relativistic collisions. The evaluation of the collision operator will be a few times more costly than

in the nonrelativistic case, firstly because five potentials need to be computed instead of two, and secondly because the kernels involve the special functions $j_{l[k]*}$ and $y_{l[k]*}$ instead of simple powers of u.

As an application of this formulation, we give in Eqs. (34) explicit forms for the diffusion and friction coefficients for an isotropic background. Finally, we calculate the electrical conductivity of an electron-ion plasma with massive ions. Our results agree with those of Spitzer and Härm[10] in the nonrelativistic limit. We also give analytical expressions for the conductivity for the limiting cases $Z \to \infty$ and $Z \to 0$ in Eqs. (41) and (42).

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Appendix A. Properties of the homogeneous solutions

In this appendix we will develop some properties of the solutions of the homogeneous radial equations (17).

1. Definitions

First we shall obtain fundamental solutions $j_{l[1]a}(z)$ and $y_{l[1]a}(z)$ to the lowest-order homogeneous differential equation

$$L_{l,a}\chi(z) = 0, \tag{A1}$$

where

$$L_{l,a}\chi(z) = (1+z^2)\frac{d^2\chi}{dz^2} + \left(\frac{2}{z} + 3z\right)\frac{d\chi}{dz} - \left(\frac{l(l+1)}{z^2} + a^2 - 1\right)\chi. \tag{A2}$$

The variable z corresponds to u/c in the main text. In this appendix, l must be an integer, a must be real, and z is in the complex plane cut along the negative real axis.

By changing the independent variable to $\gamma = \sqrt{1+z^2}$ and defining a new dependent variable $\xi(\gamma) = \sqrt{z}\chi(z)$, we obtain the equation

$$(\gamma^2-1)\frac{d^2\xi}{d\gamma^2} + 2\gamma\frac{d\xi}{d\gamma} - \left[(a+\tfrac{1}{2})(a-\tfrac{1}{2}) + \frac{(l+\frac{1}{2})^2}{\gamma^2-1}\right]\xi = 0.$$

This is the generalized Legendre equation, whose solutions are the associated Legendre functions $P^{-l-1/2}_{a-1/2}(\gamma)$ and $P^{l+1/2}_{a-1/2}(\gamma)$. We choose to define $j_{l[1]a}$ and $y_{l[1]a}$ by

$$j_{l[1]a}(z) = \sqrt{\frac{\pi}{2z}}P^{-l-1/2}_{a-1/2}(\gamma), \tag{A3}$$

$$y_{l[1]a}(z) = (-1)^{l+1}\sqrt{\frac{\pi}{2z}}P^{l+1/2}_{a-1/2}(\gamma). \tag{A4}$$

An alternative representation is obtained by Whipple's transformation (see Eq. (8.739) of Gradshteyn and Ryzhik[7])

$$j_{l[1]a}(z) = \frac{1}{z}\frac{e^{-ia\pi}}{\Gamma(a+l+1)}Q^a_l(\gamma/z),$$

$$y_{l[1]a}(z) = \frac{1}{z}\frac{(-1)^l e^{-ia\pi}}{\Gamma(a-l)}Q^a_{-l-1}(\gamma/z).$$

The Wronskian of the pair $j_{l[1]a}$ and $y_{l[1]a}$ follows from Eq. (8.741) of Gradshteyn and Ryzhik[7]; for integer l and arbitrary a it is given by

$$j_{l[1]a}\frac{d}{dz}y_{l[1]a} - y_{l[1]a}\frac{d}{dz}j_{l[1]a} = \frac{1}{\gamma z^2}. \tag{A5}$$

Therefore, the solutions $j_{l[1]*}$ and $y_{l[1]*}$ are independent. When l is a non-negative integer, then $j_{l[1]*}$ is regular at the origin while $y_{l[1]*}$ is singular.

In order to express the solutions to the higher-order equations, we introduce the functions $j_{l[k]*}(z)$ and $y_{l[k]*}(z)$ where the subscript "$*$" stands for a string of k indices. The functions $j_{l[k]*}$ are defined recursively by

$$j_{l[k+2]*aa'}(z) = \frac{j_{l[k+1]*a}(z) - j_{l[k+1]*a'}(z)}{a^2 - a'^2}; \tag{A6a}$$

the case $a = a'$ is handled by taking the limit

$$j_{l[k+2]*aa}(z) = \frac{\partial j_{l[k+1]*a}(z)}{\partial(a^2)}\,. \tag{A6b}$$

The functions $y_{l[k]*}(z)$ are defined in the same way, replacing j by y throughout. Making use of the property $(L_{l,a} - L_{l,a'})\chi = -(a^2 - a'^2)\chi$, it is readily established that $j_{l[k]*}$ and $y_{l[k]*}$ satisfy

$$L_{l,a}j_{l[k+1]*a}(z) = j_{l[k]*}(z), \tag{A7a}$$

$$L_{l,a}y_{l[k+1]*a}(z) = y_{l[k]*}(z). \tag{A7b}$$

The functions $j_{l[k]*}$ and $y_{l[k]*}$ are invariant under permutation of the indices in $*$ and invariant under change of sign of any index in $*$; these properties reflect the commutivity of $L_{l,a}$ and $L_{l,a'}$ and the symmetry $L_{l,a} = L_{l,-a}$. Also, we can write

$$y_{l[k]*} = (-1)^{l+1} j_{-l-1[k]*}, \tag{A8}$$

which reflects $L_{l,a} = L_{-l-1,a}$; and finally we have $j_{l[k]*}(-z) = (-1)^l j_{l[k]*}(z)$ and $y_{l[k]*}(-z) = (-1)^l y_{l[k]*}(z)$. In most of what follows, we list only the properties of $j_{l[k]*}$. Equation (A8) may be used to give the corresponding properties of $y_{l[k]*}$.

2. Taylor series

In the limit $z \to 0$, we can expand $P_{a-1/2}^{-l-1/2}(\gamma)$ in a Taylor series using formula (3.2.20) of Bateman[13]

$$P_\nu^\mu(\gamma) = \frac{2^\mu(\gamma^2-1)^{-\mu/2}}{\Gamma(1-\mu)} F(\tfrac{1}{2} + \tfrac{1}{2}\nu - \tfrac{1}{2}\mu, -\tfrac{1}{2}\nu - \tfrac{1}{2}\mu; 1-\mu; 1-\gamma^2),$$

where F is the hypergeometric function

$$F(a,b;c;z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \dots .$$

Substituting this into Eq. (A3), we obtain a Taylor-series expansion of $j_{l[1]a}(z)$

$$j_{l[1]a}(z) = \frac{z^l}{(2l+1)!!} F\left(\frac{l+1-a}{2}, \frac{l+1+a}{2}; l+\frac{3}{2}; -z^2\right), \tag{A9}$$

where $n!!$ denotes a double factorial

$$n!! = \begin{cases} 2^{n/2}\Gamma(\frac{1}{2}n+1), & \text{for } n \text{ non-negative and even,} \\ \sqrt{\frac{2}{\pi}}2^{n/2}\Gamma(\frac{1}{2}n+1), & \text{for } n \text{ odd.} \end{cases}$$

With this definition, we find $(2n-1)!!(-2n-1)!! = (-1)^n$.

Another useful Taylor series is obtained by a transformation of the hypergeometric function (Eq. (9.131) of Gradshteyn and Ryzhik[7])

$$F(a,b;c;z) = (1-z)^{c-a-b} F(c-a, c-b; c; z).$$

This gives

$$\frac{j_{l[1]a}(z)}{\gamma} = \frac{z^l}{(2l+1)!!} F\left(\frac{l+2-a}{2}, \frac{l+2+a}{2}; l+\frac{3}{2}; -z^2\right). \tag{A10}$$

Series (A9) and (A10) either terminate (i.e., converge everywhere) or converge inside the circle $|z| = 1$. In particular, if a is an integer and $l < |a|$, then one of the series will terminate.

It is not so easy to write down Taylor series for the higher order functions $j_{l[k]*}(z)$ ($k > 1$). However, the leading term is independent of the indices making up $*$, and can be obtained by substituting a series solution into Eqs. (A7). One finds

$$j_{l[k]*}(z) = \frac{z^{l+2k-2}}{(2k-2)!!(2l+2k-1)!!} + O(z^{l+2k}), \tag{A11a}$$

$$y_{l[k]*}(z) = \frac{(-1)^k(2l-2k+1)!!}{(2k-2)!!z^{l-2k+3}} + O(1/z^{l-2k+1}). \tag{A11b}$$

3. Asymptotic series

Since $j_{l[1]a}$ depends only on the magnitude of a, we can, without loss of generality, take $a \geq 0$ in this section. In the limit $z \to +\infty$, formula (3.2.21) of Bateman[13] may be used to derive the following asymptotic expression for $j_{l[1]a}(z)$

$$\begin{aligned} j_{l[1]a} &= (2z)^{a-1} \frac{\Gamma(a)}{\Gamma(l+a+1)} F\left(\frac{l-a+1}{2}, \frac{-l-a}{2}; -a+1; -\frac{1}{z^2}\right) \\ &+ (2z)^{-a-1} \frac{\Gamma(-a)}{\Gamma(l-a+1)} F\left(\frac{l+a+1}{2}, \frac{-l+a}{2}; a+1; -\frac{1}{z^2}\right). \end{aligned} \quad \text{(A12)}$$

This cannot be used directly when a is an integer or $l \pm a$ is a negative integer; in those cases, a limit must be taken. For a and l both non-negative integers, the leading order behavior is found to be

$$j_{l[1]a} \to \begin{cases} \dfrac{(a-1)!}{(a+l)!}(2z)^{a-1}, & \text{for } a \neq 0, \\ \dfrac{\sinh^{-1} z - \sum_{k=1}^{l} k^{-1}}{l! z}, & \text{for } a = 0, \end{cases} \quad \text{(A13)}$$

and

$$y_{l[1]a} \to \begin{cases} \dfrac{(-1)^{l+1}(a-1)!}{(a-l-1)!}(2z)^{a-1}, & \text{for } l < a, \\ \dfrac{2(a+l)!}{a!}(2z)^{-a-1}, & \text{for } l \geq a. \end{cases} \quad \text{(A14)}$$

For $0 \leq l < a$ and a an integer, $j_{l[1]a}(z)$ and $y_{l[1]a}(z)$ have the same behavior as $z \to +\infty$ (namely z^{a-1}). It is, therefore, convenient to define some combination of $j_{l[1]a}$ and $y_{l[1]a}$ in which the leading-order behavior cancels. Such a combination is

$$\begin{aligned} q_{l[1]a}(z) &= j_{l[1]a}(z) - (-1)^{l+1} \frac{(a-l-1)!}{(a+l)!} y_{l[1]a}(z) \\ &= -i\sqrt{\frac{2}{\pi z}} Q_{a-1/2}^{-l-1/2}(\gamma). \end{aligned} \quad \text{(A15)}$$

For $l < -a$, we have $q_{l[1]a} = j_{l[1]a}$; for $l \geq a$, $q_{l[1]a}$ diverges. The asymptotic behavior of $q_{l[1]a}$ for $l < a$ is

$$q_{l[1]a} \to \frac{(-1)^{l+1} 2(a-l-1)!}{a!}(2z)^{-a-1}. \quad \text{(A16)}$$

An asymptotic series for $j_{l[1]a}(z)$ valid for $l \to +\infty$ follows from formula (3.2.14) of Bateman[13]

$$j_{l[1]a}(z) = \frac{1}{(2l+1)!!}\left(\frac{2}{\gamma+1}\right)^{l+1/2} z^l F\left(-a+\frac{1}{2}, a+\frac{1}{2}; l+\frac{3}{2}; -\frac{z^2}{\gamma+1}\right).$$

The leading term is

$$j_{l[1]a}(z) \to \frac{\sqrt{\pi/2}}{\Gamma(l+3/2)} z^l (\gamma+1)^{-l-1/2}, \qquad \text{for } l \to +\infty. \tag{A17}$$

4. Recurrence relations

Recurrence relations that give $j_{l[1]a}$ in terms of $j_{l\pm1[1]a}$ and $j_{l[1]a\pm1}$ may be derived from the corresponding relations for Legendre functions[7,13]:

$$\begin{aligned} j_{l[1]a}(z) &= \frac{z}{(2l+1)\gamma}\Big[j_{l-1[1]a}(z) \\ &\qquad + (l+1-a)(l+1+a) j_{l+1[1]a}(z)\Big] && \text{(A18a)} \\ &= -\frac{1}{2a\gamma}\Big[(l-a+1) j_{l[1]a-1}(z) - (l+a+1) j_{l[1]a+1}(z)\Big] && \text{(A18b)} \\ &= \frac{1}{(l-a)\gamma}\Big[z j_{l-1[1]a}(z) - (l+1+a) j_{l[1]a+1}(z)\Big]. && \text{(A18c)} \end{aligned}$$

These relations may be combined to express $j_{l[1]a}$ in terms of any pair of its neighbors. The derivative of $j_{l[1]a}$ may be found from

$$\frac{d}{dz} j_{l[1]a}(z) = \frac{1}{\gamma} j_{l-1[1]a}(z) - \frac{l+1}{z} j_{l[1]a}(z). \tag{A19}$$

By combining Eqs. (A6) with the relations (A18a) and (A19), we can generalize these recurrence relations to multiple indices

$$\begin{aligned} j_{l[k]*}(z) &= j_{l-2[k+1]*a}(z) - (2l-1)\frac{\gamma}{z} j_{l-1[k+1]*a}(z) \\ &\qquad + (l^2 - a^2) j_{l[k+1]*a}(z), && \text{(A20)} \\ \frac{d}{dz} j_{l[k]*}(z) &= \frac{1}{\gamma} j_{l-1[k]*}(z) - \frac{l+1}{z} j_{l[k]*}(z). && \text{(A21)} \end{aligned}$$

Other recurrences involving the higher order functions may be found by differentiating Eq. (A18c) with respect to a:

$$
\begin{aligned}
&2(a+1)(l+1+a)j_{l[2]a+1,a+1}(z) \\
&\quad = 2a[zj_{l-1[2]a,a}(z) - (l-a)\gamma j_{l[2]a,a}(z)] \\
&\qquad + \gamma j_{l[1]a}(z) - j_{l[1]a+1}(z),
\end{aligned} \tag{A22a}
$$

$$
\begin{aligned}
&4(a+1)^2(l+1+a)j_{l[3]a+1,a+1,a+1}(z) \\
&\quad = 4a^2[zj_{l-1[3]a,a,a}(z) - (l-a)\gamma j_{l[3]a,a,a}(z)] + zj_{l-1[2]a,a}(z) \\
&\qquad - (l-3a)\gamma j_{l[2]a,a}(z) - (l+3a+3)j_{l[2]a+1,a+1}(z).
\end{aligned} \tag{A22b}
$$

Equations (A18) and (A19) constitute a set of second-order recurrence relations which $j_{l[1]a}$ solve. There is another independent solution which we will write as $g_{l[1]a}$. If we substitute

$$g_{l[1]a}(z) = c_{l,a} y_{l[1]a}(z)$$

into the recurrence relations, the resulting relations are equivalent to the original set provided $c_{l,a}$ satisfies

$$
\begin{aligned}
\frac{c_{l,a}}{c_{l-1,a}} &= \frac{1}{(l+a)(l-a)}, \\
\frac{c_{l,a}}{c_{l,a-1}} &= -\frac{l+1-a}{l+a}.
\end{aligned}
$$

Depending on the value of l, $c_{l,a}$ may be written in one of three forms

$$
c_{l,a} = \begin{cases}
c^{i}_{l,a} = \dfrac{(-1)^a}{(l-a)!(l+a)!}, & \text{for } l \geq a, \\
c^{ii}_{l,a} = \dfrac{(-1)^l (a-l-1)!}{(l+a)!}, & \text{for } -a \leq l < a, \\
c^{iii}_{l,a} = (-1)^a (a-l-1)!(-a-l-1)!, & \text{for } l < -a,
\end{cases} \tag{A23}
$$

where we have taken $a \geq 0$. Thus $g^{ii}_{l[1]a}(z) = c^{ii}_{l,a} y_{l[1]a}(z)$ solves the recurrence relations for $-a \leq l < a$. If this solution is extended beyond this range, it degenerates to 0 (for $l < -a$) or infinity (for $l \geq a$). The independence of $j_{l[1]a}$ and $g_{l[1]a}$ may be verified by computing the Casoratians. These may be found by substituting recurrence relations (A18) and (A19) into the Wronskian, Eq. (A5) to give

$$
\begin{aligned}
g_{l[1]a}(z) j_{l-1[1]a}(z) - j_{l[1]a}(z) g_{l-1[1]a}(z) &= -c_{l,a} \frac{1}{z^2}, \\
g_{l[1]a}(z) j_{l[1]a-1}(z) - j_{l[1]a}(z) g_{l[1]a-1}(z) &= -\frac{c_{l,a}}{l+1-a} \frac{1}{z}.
\end{aligned}
$$

The function $q_{l[1]a}(z)$ defined in Eq. (A15) may be written as $j_{l[1]a}(z) - g^{ii}_{l[1]a}(z)$. Thus $q_{l[1]a}(z)$ solves the recurrence relations for $l < a$, and this gives a solution independent of $j_{l[1]a}$ for $-a \le l < a$.

5. Generating function

From the differential recurrence, Eq. (A21), we obtain

$$f_{l+m[k]*}(\gamma) = (-1)^m \frac{d^m}{d\gamma^m} f_{l[k]*}(\gamma),$$

where

$$f_{l[k]*}(\gamma) = \frac{y_{l[k]*}(\sqrt{\gamma^2 - 1})}{(\gamma^2 - 1)^{l/2}}.$$

Performing a Taylor-series expansion of $f_{0[k]*}(\gamma - \epsilon z)$ gives

$$f_{0[k]*}(\gamma - \epsilon z) = \sum_{l=0}^{\infty} f_{l[k]*}(\gamma) \frac{(\epsilon z)^l}{l!}.$$

This leads to the following generating function

$$y_{0[k]*}\Big(\sqrt{z^2(1+\epsilon^2) - 2\epsilon z\gamma}\Big) = \sum_{l=0}^{\infty} y_{l[k]*}(z) \frac{\epsilon^l}{l!}. \tag{A24}$$

6. Special cases

Simple closed expressions exist in certain special cases (in the following equations, $\sigma = \sinh^{-1} z$):

$$j_{a[1]a}(z) = \frac{1}{(2a-1)!!z^{a+1}} \int_0^z \frac{z'^{2a}}{\gamma'}\, dz', \tag{A25a}$$

$$j_{a-1[1]a}(z) = \frac{z^{a-1}}{(2a-1)!!}, \tag{A25b}$$

$$j_{0[1]a}(z) = \frac{\sinh(a\sigma)}{az} = \frac{(\gamma+z)^a - (\gamma+z)^{-a}}{2az}, \tag{A25c}$$

$$j_{-1[1]a}(z) = \frac{\cosh(a\sigma)}{z} = \frac{(\gamma+z)^a + (\gamma+z)^{-a}}{2z}, \tag{A25d}$$

$$j_{-a-1[1]a}(z) = (-1)^a \frac{(2a-1)!!}{z^{a+1}}, \tag{A25e}$$

$$j_{0[k]0\ldots0}(z) = \frac{\sigma^{2k-1}}{(2k-1)!z}, \tag{A25f}$$

where we have taken $a \geq 0$.

The recursion relations, Eqs. (A18), (A20), and (A22), together with Eq. (A25f) allow $j_{l[k]*}$ for $0 \leq k \leq 3$ for integer l and integer indices $*$ to be expressed in terms of elementary functions. The multiple-index homogeneous solutions that we need can be expressed simply in terms of the single-index solutions:

$$j_{l[2]02}(z) = \frac{z}{2} j_{l+1[1]1}, \tag{A26a}$$

$$j_{l[2]11}(z) = \frac{z}{2} j_{l+1[1]0}, \tag{A26b}$$

$$j_{l[2]22}(z) = \frac{z}{2} j_{l+1[1]1} + \frac{z^2}{2} j_{l+2[1]0}, \tag{A26c}$$

$$j_{l[3]022}(z) = \frac{z^2}{8} j_{l+2[1]0}. \tag{A26d}$$

For definiteness, we catalog all the required functions $j_{l[k]*}(z)$ and $y_{l[k]*}(z)$ for $l = 0$ and 1:

$$j_{0[1]0} = \frac{\sigma}{z}, \qquad y_{0[1]0} = -\frac{1}{z}, \tag{A27a}$$

$$j_{0[1]1} = 1, \qquad y_{0[1]1} = -\frac{\gamma}{z}, \tag{A27b}$$

$$j_{0[1]2} = \gamma, \qquad y_{0[1]2} = -\frac{1 + 2z^2}{z}, \tag{A27c}$$

$$j_{0[2]02} = \frac{z\gamma - \sigma}{4z}, \qquad y_{0[2]02} = -\frac{z}{2}, \tag{A27d}$$

$$j_{0[2]11} = \frac{\gamma\sigma - z}{2z}, \qquad y_{0[2]11} = -\frac{\sigma}{2}, \tag{A27e}$$

$$j_{0[2]22} = \frac{-z\gamma + (1 + 2z^2)\sigma}{8z}, \qquad y_{0[2]22} = -\frac{\gamma\sigma}{2}, \tag{A27f}$$

$$j_{0[3]022} = \frac{-3z\gamma + (3 + 2z^2)\sigma}{32z}, \qquad y_{0[3]022} = \frac{-\gamma\sigma + z}{8}. \tag{A27g}$$

$$j_{1[1]0} = \frac{\gamma\sigma - z}{z^2}, \qquad y_{1[1]0} = -\frac{\gamma}{z^2}, \tag{A28a}$$

$$j_{1[1]1} = \frac{z\gamma - \sigma}{2z^2}, \qquad y_{1[1]1} = -\frac{1}{z^2}, \tag{A28b}$$

$$j_{1[1]2} = \frac{z}{3}, \qquad y_{1[1]2} = -\frac{(1 - 2z^2)\gamma}{z^2}, \tag{A28c}$$

$$j_{1[2]02} = \frac{-3\gamma\sigma + (3z + z^3)}{12z^2}, \qquad y_{1[2]02} = \frac{\gamma}{2}, \tag{A28d}$$

$$j_{1[2]11} = \frac{-3z\gamma + (3 + 2z^2)\sigma}{8z^2}, \qquad y_{1[2]11} = \frac{1}{2}, \tag{A28e}$$

$$j_{1[2]22} = \frac{-(3 - 6z^2)\gamma\sigma + (3z - 5z^3)}{72z^2}, \qquad y_{1[2]22} = \frac{\gamma + z\sigma}{2}, \tag{A28f}$$

$$j_{1[3]022} = \frac{(15 + 6z^2)\gamma\sigma - (15z + 11z^3)}{288z^2}, \qquad y_{1[3]022} = \frac{z\sigma}{8}. \tag{A28g}$$

The function $y_{0[k]*}(z)$ is the kernel that occurs in Eq. (10).

7. Numerical methods

Our final concern is to determine a method by which $j_{l[k]*}$, $y_{l[k]*}$, and their derivatives may be calculated accurately and quickly. A direct use of the analytic forms is ill-advised, particularly when z is small and l is large. For example, consider the numerator of the analytic form for $j_{1[2]02}$ given in Eq. (A28d) in the limit $z \to 0$. This consists of three terms, the largest of which is proportional to z; however, the sum is proportional to z^5.

Given $j_{l[1]a}$, Eqs. (A26) may be used to calculate the required multiple-index solutions. Furthermore Eq. (A21) may be used to give the derivatives of $j_{l[k]*}$. Equation (A4) gives $y_{l[k]*}$ in terms of $j_{l[k]*}$. Thus the problem is reduced to calculating $j_{l[1]a}(z)$ of all integer $-L-1 \le l \le L$, integer $a \ge 0$, and real $z \ge 0$. (For $a < 0$, we can use $j_{l[1]-a}(z) = j_{l[1]a}(z)$. For $z < 0$, we can use $j_{l[1]a}(-z) = (-1)^l j_{l[1]a}(z)$.)

We will also be able to avoid problems with numerical underflow and overflow by computing $\tilde{\jmath}_{l,a}$ where

$$j_{l[1]a}(z) = \frac{z^l}{(2l+1)!!}\tilde{\jmath}_{l,a}(z).$$

In the nonrelativistic limit $z \to 0$, we have, from Eq. (A9), $\tilde{\jmath}_{l,a} \to 1$.

Our main tool for calculating $j_{l[1]a}$ will be the recurrence relation (A18a) which when written in terms of $\tilde{\jmath}_{l,a}$ becomes

$$\tilde{\jmath}_{l-2,a} = \gamma\tilde{\jmath}_{l-1,a} - \frac{(l-a)(l+a)}{(2l-1)(2l+1)}z^2\tilde{\jmath}_{l,a}. \tag{A29}$$

In order to apply it we need to examine the stability of the recurrence for large l. In this limit, the recurrence relation is approximately

$$\tilde{\jmath}_{l-2,a} \approx \gamma \tilde{\jmath}_{l-1,a} - \tfrac{1}{4} z^2 \tilde{\jmath}_{l,a},$$

whose solution is

$$\frac{\tilde{\jmath}_{l,a}}{\tilde{\jmath}_{l-1,a}} \approx \frac{2}{\gamma \pm 1}.$$

Comparing this with the leading term in the asymptotic series (A17) shows that the solution we want corresponds to the upper sign. This solution is dominant when the recurrence relation is applied in the backwards direction. If we start with large l with some arbitrary mixture of the dominant and subdominant solutions, then on each application of the backwards recurrence relation, the subdominant solution decreases by $(\gamma - 1)/(\gamma + 1) = z^2/(\gamma + 1)^2$ relative to the desired solution.

If we desire to compute $\tilde{\jmath}_{l,a}$ to accuracy δ for $l \le L$, we choose an L' such that

$$L' > L + \frac{\log(\delta)}{2 \log(z/(\gamma + 1))}.$$

We set $\tilde{\jmath}'_{L',a}$ and $\tilde{\jmath}'_{L'-1,a}$ so that their ratio is given by $\tilde{\jmath}'_{L',a}/\tilde{\jmath}'_{L'-1,a} = 2/(\gamma + 1)$. We then use Eq. (A29) as a backwards recurrence to give $\tilde{\jmath}'_{l,a}$ for $0 \le l \le L$. At this point $\tilde{\jmath}'_{l,a}$ differs from the desired solution only by an overall multiplicative factor. This may be determined from Eq. (A25b) which gives $\tilde{\jmath}_{a-1,a} = 1$. Thus $\tilde{\jmath}_{l,a} = \tilde{\jmath}'_{l,a}/\tilde{\jmath}'_{a-1,a}$. Because of the degeneracy in Eq. (A29) the values of $\tilde{\jmath}_{l,a}$ for $0 \le l < a$ are independent of the choice of L' and starting values $\tilde{\jmath}'_{L',a}$ and $\tilde{\jmath}'_{L'-1,a}$. The recurrence is effectively restarted at $l = a - 1$.

Various optimizations to this scheme are possible. For example, it is only necessary to start the recursion at $l = L'$ for one value of a, e.g., $a = 0$. For other values of a we can start the recursion at $l = L$ by rewriting Eq. (A18c) as

$$\tilde{\jmath}_{l,a} = \frac{(2l + 1)\tilde{\jmath}_{l-1,a-1} - (l + 1 - a)\gamma \tilde{\jmath}_{l,a-1}}{l + a}, \tag{A30}$$

and using this recurrence to give $\tilde{\jmath}_{L,a}$ and $\tilde{\jmath}_{L-1,a}$ in terms of $\tilde{\jmath}_{L,a-1}$, $\tilde{\jmath}_{L-1,a-1}$, and $\tilde{\jmath}_{L-2,a-1}$.

For large values of z, L' becomes large because the behavior of the dominant and subdominant solutions is nearly the same. It is, therefore, possible to use forward recursion using Eq. (A29) to obtain $\tilde{\jmath}_{l,a}$ for $l > a$. Starting values are given by $\tilde{\jmath}_{a-1,a} = 1$ and $\tilde{\jmath}_{a,a}$ which may be calculated using $\tilde{\jmath}_{0,0} = \sigma/z$ and recurrence relation (A30). As before, backwards recursion should be used for $0 \le l < a$.

For $l < -a$, we can compute $\tilde{\jmath}_{l,a}$ by backwards recursion using Eq. (A29) together with the starting value $\tilde{\jmath}_{-a-1,a} = 1$. For $-a \le l < 0$, we could continue the backwards recursion using as starting values $\tilde{\jmath}_{0,a}$ and $\tilde{\jmath}_{1,a}$ as found above. However, it is sometimes useful to be able to compute $q_{l[1]a}$; but this cannot be accurately computed using Eq. (A15) when z is large because of the large cancellation that occurs in this limit. Instead, we compute $q_{l[1]a}$ directly. To avoid problems with underflow and overflow, we work with $\tilde{q}_{l,a}$ which is defined by

$$q_{l[1]a}(z) = \frac{z^l}{(2l+1)!!}\tilde{q}_{l,a}(z).$$

We have seen that $q_{l[1]a}$ satisfies the same recurrence relations as $j_{l[1]a}$. This implies that $\tilde{q}_{l,a}$ satisfies the recurrence relation (A29). From Eqs. (A25c) and (A25d), together with the recurrence relation, we have

$$\tilde{q}_{-1,a}(z) = \frac{1}{(\gamma+z)^a}, \tag{A31a}$$

$$\tilde{q}_{-2,a}(z) = \frac{\gamma+az}{(\gamma+z)^a}. \tag{A31b}$$

We can then utilize backwards recursion using Eq. (A29) to give $\tilde{q}_{l,a}$ for all $-a \le l < 0$. Finally, we can compute $\tilde{\jmath}_{l,a}$ for $-a \le l < 0$ using Eq. (A15) which gives

$$\tilde{\jmath}_{l,a}(z) = \tilde{q}_{l,a}(z) + \frac{(a-l-1)!}{(a+l)!}\frac{(-1)^{l+1}z^{-2l-1}}{(-2l-3)!!(-2l-1)!!}\tilde{\jmath}_{-l-1,a}. \tag{A32}$$

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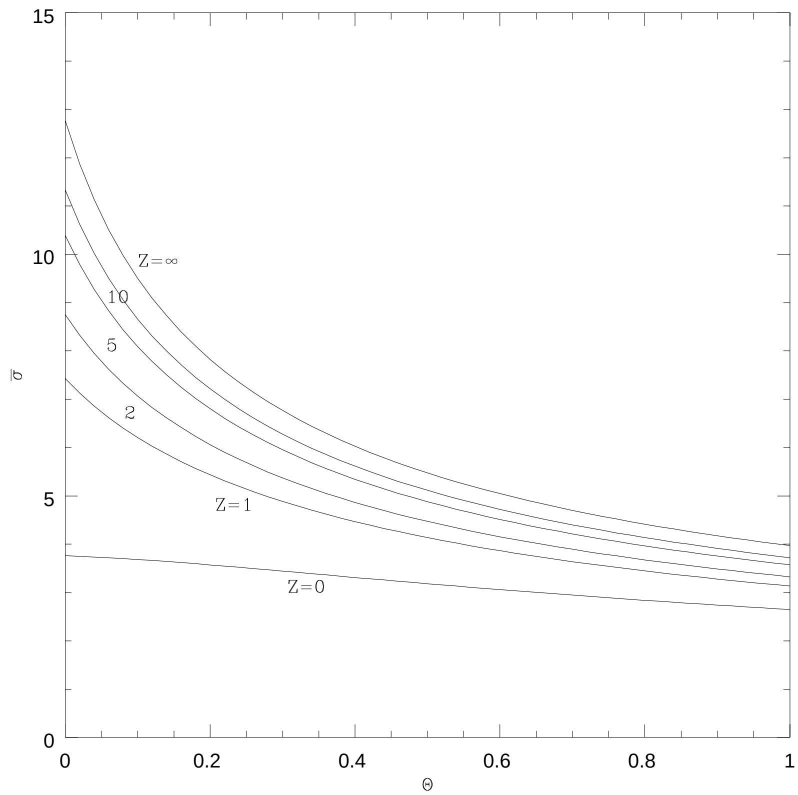
Table

TABLE I. Conductivities for various values of the normalized temperature $\Theta = T_e/(511\,\mathrm{keV})$ and the effective ion charge state Z. The conductivities are normalized to $(4\pi\epsilon_0^2)/(m_e^{1/2}q_e^2 \log\Lambda^{e/e})(T_e^{3/2}/Z)$.

Θ \ Z	0	1	2	5	10	∞
0	3.75994	7.42898	8.75460	10.39122	11.33006	12.76615
0.01	3.75490	7.27359	8.53281	10.07781	10.95869	12.29716
0.02	3.74920	7.12772	8.32655	9.78962	10.61952	11.87371
0.05	3.72852	6.73805	7.78445	9.04621	9.75405	10.81201
0.1	3.68420	6.20946	7.06892	8.09361	8.66306	9.50746
0.2	3.57129	5.43667	6.06243	6.80431	7.21564	7.82693
0.5	3.18206	4.13733	4.47244	4.88050	5.11377	5.47602
1	2.65006	3.13472	3.32611	3.57303	3.72206	3.96944
2	2.03127	2.27862	2.39205	2.54842	2.64827	2.82473
5	1.33009	1.45375	1.51805	1.61157	1.67382	1.78870
10	0.94648	1.02875	1.07308	1.13856	1.18263	1.26490
20	0.67042	0.72743	0.75853	0.80472	0.83593	0.89443
50	0.42422	0.46003	0.47965	0.50885	0.52861	0.56569
100	0.29999	0.32528	0.33915	0.35979	0.37377	0.40000



Figure

FIG. 1. The normalized conductivity as a function of the normalized temperature $\Theta = T_e/(511\,\mathrm{keV})$ for various values of the effective ion charge state Z.